Arrival Times in Quantum Mechanics and time-energy uncertainty relation

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1 Introduction

The notion of time in quantum mechanics has, up to now, been controversial and pending due to its two-fold status. Firstly, time enters Schrödinger equation not as an observable (like what position behaves) but rather as an external parameter. The measurement result and probability distribution of a quantity in a quantum state are described in terms of the instant of time. Secondly, we may, in principle, ask for the instant of time at which a certain quantity takes an initial value and another instant of time while the system evolves and the quantity takes a final value. In this sense we are, in essence, to measure the value of time in terms of physical systems undergoing changes and accordingly the instant of time displays the character of a dynamical measurable quantity. Examples of this are the concepts of arrival times, dwell times, lifetimes and generally the times required to find a dynamical variable with a given value. Since the instant of time can be regarded as an observable, it naturally raises the question that whether we can include it into the quantum formulation.

The standard quantum formalism of measurable quantities is to find a suitable quantum operator through the correspondence principle, e.g., starting from the corresponding classical expression of a certain quantity and quantizing the quantity by using specific quantization rules. A most common way of incorporating classical quantities into quantum formulation is the canonnical quantization method[1], i.e., to replace the Poisson's Bracket of a pair of canonical variables with commutation bracket of corresponding operators. Given the Hamiltonian $H(q,p)$ of a conservative classical system (with no explicit dependence on t), we can always make a canonical transformation of (q,p) to new canonical variables (H,T), where H is the Hamiltonian of the system and T its conjugate variable, which satisfies Hamilton's equation[2][3]

$$\frac{dT}{dt} = \{H,T\} = \frac{\delta H}{\delta H}\frac{\delta T}{\delta T} - \frac{\delta T}{\delta H}\frac{\delta H}{\delta T} = 1. \tag{1.1}$$

The above equation indicates that T is exactly the interval of time. By taking the quantization, $\{H,T\} \to 1/i\hbar[\hat{H},\hat{T}]$, where we postulate that both

$\hat{H}$ and $\hat{T}$ are self-adjoint operators, we reach the quantum commutation relation,

$$[\hat{H}, \hat{T}] = i\hbar, \tag{1.2}$$

which is true in both Heisenberg picture and Schroödinger picture, thus yields the uncertainty relation,

$$\Delta H \Delta T \geq 1/2 \, |\langle [\hat{H}, \hat{T}] \rangle|, \tag{1.3}$$

where ΔH and ΔT are the root-mean-square deviations of H and T respectively.

For the time being, we have not verify whether there exists such a self-adjoint operator for the instant of time. Unfortunately, Pauli[4] pointed out that the existence of a self-adjoint time operator is incompatible with the bounded or semibounded character of the Hamiltonian spectrum. The argument runs as follows: if there exists such a self-adjoint time operator $\hat{T}$ and correspondingly the canonical conjugation relation $[\hat{H}, \hat{T}] = i\hbar$ holds, the application of the unitary operator $\exp(iE_0\hat{T}/\hbar)$ to the energy eigenstate $|E\rangle$ then yields a new energy eigenstate with energy eigenvalue $E - E_0$, which implies the spectrum of the Hamiltonian can be extended continuously over the whole axis $[-\infty, \ \infty]$.

The nonexistence of a self-ajoint time operator has resulted in a series of consequences. Firstly, unlike the complementary quantities of position and momentum, there is no such thing as uncertainty relation for time and energy. Even the meaning of the time spread ΔT has still remained unclear up to now. Secondly, the formal formulation of traversal, tunneling times as well as other times involved in a system as a dynamical variable are yet subject to debate. One of the simplest examples is that associated with the time of a free point-like particle arriving at a position. Such arrival time is well-defined and can be easily solved from the relevant classical equation of motion, and its probability distribution can be determined within arbitrary precision; however in quantum level such a probability distribution cannot be achieved.

Many attempts have been made to overcome Pauli's argument. There are mainly three approaches. The first one is interested in mathematical investigation into the argument itself. As usually quoted, it is not a rigorous theorem but a formal argument. Galapon emphasized that no mathematical rigor was found in the original formulation, especially no serious study was undertaken on the domains of the operators involved. He has shown that it is consistent to assume a bounded, self-adjoint time operator conjugate to a Hamiltonian with an unbounded, semibounded, or finitely countable point spectrum[5][6][7][8][9]. In any case, it is unnecessary to impose the requirement of self-ajointness in a formulation of quantum mechanics that connects observables with positive operator-valued measures(POVMs), i.e., with nonorthogonal resolutions of the identity.

The second approach focuses on operational understanding of arrival times and arrival time distributions without considering time operators. However there is a major problem that has led to a series of controversies: to which extent is the measured arrival time independent of the experimental device? Allcock[10] drew a negative conclusion after studying on the absorption rate of his imaginary-potential models. He argued that it is impossible to construct any operationally meaningful and device-independent probability formulation. Muga *et al.*, however, constructed complex-valued potentials that perfectly absorb at a given wave number in an arbitrary small spatial interval[11]. Later on, Aharonov $etal.$[12] constructed several toy models to measure arrival-time distributions and he agreed with Allcock that the more accurate the measurement is, the more distorted the distribution becomes.

The third approach endeavors to incorporate the observable of time into quantum formulation by defining proper arrival-time operators, arrival-time distributions and average arrival times. Aharonov and Bohm[13] introduced a time-of-arrival operator by a quantization of the classical expression which is not self-adjoint but maximally symmetric. Later, Kijowski[14] obtained an arrival-time distribution with minimal variance by imposing a set of intuitive axioms based on classical considerations. Finally, Giannitrapani[15] showed that the positive operator valued measure[16] associated with the Aharonov-Bohm time operator leads to Kijowski's distribution. This is the customary approach to arrival times of freely moving particles in the current literature.

The expressions reviewed in this paper are referred to as ideal within this work. This means that they rely only on theoretical considerations without any relations to actual experiments or measurement situations.

This work consists mainly of two parts: arrival time and time-energy uncertainty relation. Its organization is as follows.

First we introduce the controversial role of time in quantum theory and the obstacle that prevents time from being formulated into quantum formalism. This leads to a variety of approaches to overcome the problem, in which the notion of arrival time plays a central role. Thus we start from the arrival-time distribution in classical formulation and then go into the introduction of quantum arrival time distribution, wherein we point out the lack of positivity in quantum flux. Hence we present some ideas of constructing a positive distribution, e.g., Kijowski's distribution, Aharonov-Bohm operator and POVMs, Halliwell-Yearsley distribution, as well as some connections between different approaches. After that we continue to discuss another problem concerning the role of time–how to interpret time in time-energy uncertainty relation. In this part, different approaches, such as Mandelstam-Tamm uncertainty relation and Bohr-Wigner uncertainty relation, are presented in detail.

2 Classical arrival time

Before considering quantum arrival time it is necessary first to understand the arrival-time distributions of the ensemble of classical particles. Here we assume that, for simplicity, the classical particle considered is structureless and moving along a straight line, i.e., one dimension space with its position denoted as x and momentum p. Since the case of interacting particles and multiple crossings is more or less difficult to understand even in classical mechanics, we restrict our interest in deterministic motion, which is described by Liouville's equation.

In the simplest case a classical particle may pass a definite spatial point x once and only once provided that there are neither external potential barrier nor interactions between particles that result in the change of the trajectories; otherwise the trajectory of the particle might cross x several times. The first crossing yields the first passage time and likewise the n-th crossing yields the n-th passage time. There exists a distribution of times for the n-th passage of noninteracting particles.

First we consider the free motion case. In classical mechanics, the classical ensemble of freely moving rightward noninteracting particles is described by the phase-space distribution function $\varrho(x, p; t)$, that is normalized to one and satisfies $\varrho(x, q \leq 0; t) = 0$. The arrival time at $x = x_A$ of a freely rightward moving noninteracting particles with initial position $x_0 < x_A$ and initial momentum $p_0 > 0$ is given by

$$t_A = \frac{m(x_A - x_0)}{p_0}. \tag{2.1}$$

For free motion case the particle passes the point x_A once and only once, therefore Equation (2.1) also represents the first arrival time. The distribution of first arrival times is described by the probability flux or current density, $J(x, t)$,

$$J(x, t) = \int_{-\infty}^{\infty} \mathrm{d}x \int_{0}^{\infty} \mathrm{d}p \varrho(x, p; t) \frac{p}{m} \delta(x - x_A), \tag{2.2}$$

which is the mathematical expectation of the phase-space function

$$F(x, p; t) = \frac{p}{m} \delta(x - x_A). \tag{2.3}$$

Using equation (2.1), the average arrival time is given for free motion by

$$\langle t_A \rangle = \int_{0}^{\infty} J(x, t) t \mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_{0}^{\infty} \mathrm{d}p_0 \varrho(x_0, p_0; 0) \frac{m(x_0 - x_A)}{p_0}. \tag{2.4}$$

The integral exists if $\varrho(x_0, p_0; 0)/p_0$ is integrable at $p_0 = 0$, that is to say, ϱ at least satisfies the asymptotic condition $\varrho \sim p_0^{\epsilon}, \epsilon > 0$, when $p_0 \to 0$.

For an ensemble of free particles with momenta of arbitrary sign, the positive flux, given by

$$J_+(x_A,t) = \int_0^\infty \mathrm{d}p \varrho(x_A,p;t)\frac{p}{m}, \tag{2.5}$$

minus the negative flux, given by

$$J_-(x_A,t) = \int_{-\infty}^0 \mathrm{d}p \varrho(x_A,p;t)\frac{p}{m}, \tag{2.6}$$

gives the total arrival-time distribution[17],

$$\begin{aligned} J_{tot} = J_+(x_A,t) - J_-(x_A,t) &= \int_{-\infty}^\infty \mathrm{d}p \varrho(x_A,p;t)\frac{|p|}{m} \\ &= \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty \mathrm{d}p \varrho(x,p;t) F_{tot}(x,p;x_A), \end{aligned} \tag{2.7}$$

which corresponds to the ensemble average of the phase space function

$$F_{tot}(x,p;x_A) = \frac{|p|}{m}\delta(x-x_A). \tag{2.8}$$

When the ensemble contains particles interacting with external potential or contains more complex dynamics, multiple crossings are possible, and J_{tot} is no longer the first passage time distribution. Assume that all particles are initially on the left side and that any particle that crosses x is eliminated at the crossing instant due to absorbing boundary conditions[17]. The absorbing boundary condition is given by

$$\lim_{\epsilon \to 0+} \varrho(x_A - \epsilon, p; t) = 0, \;\; p < 0, \tag{2.9}$$

and the probability flux

$$J_{abs}(x_A,t) = -\frac{\mathrm{d}N(t)}{\mathrm{d}t} \tag{2.10}$$

provides the first passage time distribution, where

$$N(t) = \int_{-\infty}^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}p \varrho(x,p;t) \tag{2.11}$$

is the time-dependent diminishing norm which depends on x_A. If not all particles eventually reach the absorbing point x_A, $J_{tot}(x_A, t)$ is not normalized to 1 but it can be normalized by dividing Eq.(2.10) by the total norm absorbed, $1 - N(\infty) = \int \mathrm{d}t J_{abs}(x_A, t)$.

According to stochastic diffusive process, the passage events at point x_A can be classified into two types: one is to arrive at x_A directly via trajectory which is restricted in the left subspace (suppose all particles starts from the initial point $x = x_0 < 0$ at $t = 0$); the other is to cross x_A for more than once. Denote by $P_0(x, x_0; t)$ the probability density for the former case, it can be calculated by solving the stochastic diffusive equation with absorbing boundary conditions at x_A. The probability density for the latter case can then be expressed as

$$P(x, x_0; t) = P_0(x, x_0; t) + \int_0^t f_{x_A, x_0}(s) P(x, x_A; t - s) ds, \tag{2.12}$$

where $f_{x_A,x_0}(s)$ is the first passage time distribution and $P(x, x_A; t - s)$ is the probability density for the case of having crossed x_A at some instant $s < t$ and crossing x_A again at instant t. This is the *renewal equation* in classical stochastic diffusive process. As can be easily seen in Eq.(2.12), the probability density for being at instant t can be decomposed into that for the events of arriving at x_A directly and that for the events of having crossed x_A at least once before the crossing instant t. $f_{x_A,x_0}(s)$ is the probability flux at x_A with an absorbing boundary, but it can also be obtained by solving the integral equation by Laplace transform.

3 Quantum arrival times for free motion

The incorporation of arrival time as a quantum observable into quantum formulation has turned out to be not an easy task. There has been many proposals and arguments presented during the long history. Of all the attempts made to investigate into this topic, there are mainly two distinct types of approaches. One is to define an arrival-time operator and the other is to focus on the definition of arrival-time distributions or average arrival time for quantum mechanics without dealing with the problem of finding a proper arrival-time operator.

As is mentioned above in the *introduction* section, Pauli has pointed out that there exists an inevitable contradiction between the existence of a self-ajoint time operator and Hamiltonians with bounded or semibounded energy spectrum. Since then many efforts haven been made to overcome this problem concerned with the case of arrival times.

In 1961 Bohm and Aharonov[13] introduced a non-self-adjoint arrival-time operator $\widehat{T}_{AB}$ by quantizing and symmetrizing the classical first-arrival-time Eq.(2.1),

$$\widehat{T}_{AB} = \frac{m}{2}\left(\widehat{x}\frac{1}{\widehat{p}} + \frac{1}{\widehat{p}}\widehat{x}\right). \tag{3.1}$$

The operator is maximally symmetric and exhibits the characteristics most closed to what we desire of a self-adjoint operator. It is conjugate to the Hamiltonian and thus implies an uncertainty relation for time and energy. In 1996, Grot *et al.*[18] defined a self-adjoint variant of $\widehat{T}_{AB}$ which circumvents Pauli's argument by modifying the commutation relation $[\widehat{H}, \widehat{T}] = i\hbar$. Later, Galapon[5][7] considered self-adjoint arrival-time operators for spatially confined particle. Some other important contributions are due to Razavi[19], Kijowski[20] and Delgado and Muga[21].

On the other hand, many authors tried to find an arrival-time distribution for quantum free particles. Allcock studied an arrival-time distribution based on a simplified detection procedure by means of an absorbing potential. His conclusion, however, was a negative one. He claimed that a detector-independent formulation cannot be found, but he proposed to obtain an ideal arrival-time distribution by a deconvolution of the absorption rate with the apparatus response. Kijowski presented a quantum arrival-time distribution $\Pi(t;\psi(t_0)) = \sum_\alpha |\langle t,\alpha|\psi(t_0)\rangle|^2$ by translating the classical arrival-time distribution $\Pi(t;f)$ into the corresponding quantum version. A more general treatment based on Kijowski's work has been obtained by Werner[22]. Further contributions to arrival-time distributions are due to Yamada and

Takaki[23][24][25] and due to Kochanski and Wodkiewicz[26]. A treatment in the framework of Bohmian mechanics has been proposed by Leavens[27][28] and other authors[29]. Recently, Ruseckas and Kaulakys employed the notion of weak measurements to define arrival-time distributions[30][31].

An important contribution has been made by Giannitrapani[15], who used the concept of positive operator valued measures(POVMs) to revealed the connection between the Aharonov-Bohm arrival-time operator and Kijowski's distribution. This distribution for arrival times is widely accepted.

4 Quantum mechanical flux

The classical probability flux $J(x,t)$ of Eq. (2.2) is understood as the arrival-time distribution for positive-momenta particles. In quantum mechanics, as we know, the probability flux of the state $|\psi_t\rangle$, however, takes a different form,

$$J^{\mathrm{QM}}(x,t) = \frac{\hbar}{m}\mathrm{Im}\left(\overline{\psi(x,t)}\frac{\partial}{\partial x}\psi(x,t)\right), \tag{4.1}$$

which does not seem to be directly associated with the classical flux by the quantization rule. The real problem of the quantum mechanical flux lies in that there is the possibility that $J^{\mathrm{QM}}(x,t)$ may be negative over a long, but finite time interval[35]. This is called the backflow effect. Such non-positiveness indicates that the quantum mechanical flux $J^{\mathrm{QM}}(x,t)$ does not represent a true arrival-time distribution. It is nonetheless a well-defined form of a probability distribution for arrival times due to its easily calculable expression.

As for the total probability flux J_{tot} of Eq. (2.7), more problems emerge when considering the corresponding quantum version. Notice that $\Theta(\pm\widehat{p})$ do not commute with the quantum probability flux operator J^{QM}, it follows that J_+ and J_- in Eq. (2.5) and Eq. (2.6) respectively, which are the average of the phase-space functions

$$F_+(x,p) = \delta(x-x_A)\Theta(p)p/m \tag{4.2}$$

and

$$F_{-}(x,p) = \delta(x - x_A)\Theta(-p)p/m, \tag{4.3}$$

do not provide a unique quantization formalism but imply the infinite possibilities of symmetrization. Many of the so-called positive probability flux operators derived in this way fail to display positiveness since there is the backflow effect. Moreover, the total quantum flux operators cannot be clearly decomposed into the left-sided and right-sided components like their classical counterpart.

A positive definite expression is put forward for the arrival-time distribution for those particles which pass the arrival point x_A in Bohm trajectory approach[27],

$$J_{\mathrm{B}}(x_A,t) = \frac{|J(x_A,t)|}{\int_{-\infty}^{\infty} \mathrm{d}t |J(x_A,t)|}. \tag{4.4}$$

This distribution flux composes of the left-sided and right-sided parts but the justification of its expression still remains unknown until now.

5 Kijowski's distribution

In an important paper on time operator and time-energy uncertainty principle, Kijowski proposed an axiom approach for arrival-time distributions for free particles and introduced a quantum probability density on the basis of classical correspondence.

Let $\mathcal{D}$ be the space consisting of normalized, absolute continuous square integrable wave functions with support restricted to positive momenta, i.e., $\mathcal{D} = \{\psi \in L^2(\mathbb{R}),\ \tilde{\psi}(k) = 0,\ \text{for} k < 0,\ ||\psi_t||^2 = 1\}$, and $\widehat{\Pi}$ an operator on the space $\mathcal{D}$. The bilinear form $\Pi[\psi_t] := \langle\psi_t|\widehat{\Pi}|\psi_t\rangle$ will display the characteristic of a quantum distribution for arrival times with ψ_t belonging to the domain of $\widehat{\Pi}$. Analogous to the classical case[14], Kijowski postulates the following conditions:

(i) $\Pi[\psi_t] = \langle\psi_t|\widehat{\Pi}|\psi_t\rangle \geq 0$;

(ii) $\int_{-\infty}^{\infty} \mathrm{d}t\langle\psi_t|\widehat{\Pi}|\psi_t\rangle = 1$;

(iii) Let $\widehat{\pi}$ be the unitary parity operator and $\widehat{\tau}$ the anti-unitary operator of time reverse. One has

$$\langle k|\widehat{\tau}\widehat{\pi}|\psi_t\rangle = \overline{\tilde{\psi}_t(k)} \tag{5.1}$$

and

$$\langle\psi_0|\widehat{\Pi}|\psi_0\rangle = \langle\widehat{\tau}\widehat{\pi}\psi_t|\widehat{\Pi}|\widehat{\tau}\widehat{\pi}\psi_t\rangle, \tag{5.2}$$

or, equivalently, denoted as

$$\Pi[\tilde{\psi}_0] = \Pi[\overline{\tilde{\psi}_0}]. \tag{5.3}$$

In particular, an operator fulfilling (i)-(iii) is a quantum version of the classical phase-space function $F(x;p;0)$, Eq. (2.3),

$$\widehat{\Pi}_K = \frac{1}{m}\widehat{p}^{1/2}\delta(\widehat{x})\widehat{p}^{1/2}. \tag{5.4}$$

Note, however, that the quantum mechanical flux Eq. (4.1) does not fulfil condition (i). The quantum version of Kijowski's theorem then reads:

Theorem 5.1 (Kijowski) *Let $\widehat{\Pi}$ be an operator fulfilling conditions* (i)-(iii) *and satisfies*

$$\int_{-\infty}^{\infty} \mathrm{d}t\langle\psi_t|\widehat{\Pi}|\psi_t\rangle t^2 < \infty. \tag{5.5}$$

Define the mean values

$$\bar{t}_{\widehat{\Pi}} = \int_{-\infty}^{\infty} \mathrm{d}t\langle\psi_t|\widehat{\Pi}|\psi_t\rangle t \tag{5.6}$$

and

$$\bar{t}_K = \int_{-\infty}^{\infty} \mathrm{d}t\langle\psi_t|\widehat{\Pi}_K|\psi_t\rangle t, \tag{5.7}$$

then:

$$\text{(1)}\quad \bar{t}_{\widehat{\Pi}} = \bar{t}_K \tag{5.8}$$

$$\text{(2)}\quad \int_{-\infty}^{\infty} \mathrm{d}t\langle\psi_t|\widehat{\Pi}|\psi_t\rangle(t-\bar{t}_{\widehat{\Pi}}) \geq \int_{-\infty}^{\infty} \mathrm{d}t\langle\psi_t|\widehat{K}|\psi_t\rangle(t-\bar{t}_K). \tag{5.9}$$

In addition, if equality in (2) holds for any $\psi \in \mathcal{D}$, then we have $\langle\psi_t|\widehat{\Pi}|\psi_t\rangle = \langle\psi_t|\widehat{\Pi}_K|\psi_t\rangle$.

The distribution function which has the minimal variance on all possible distributions fulfilling conditions (i)-(iii) is Kijowski's distribution, which is given by

$$\Pi_K(x_A=0,t)=\langle\psi_t|\widehat{\Pi}_K|\psi_t\rangle=\left|\int_0^\infty \mathrm{d}p\sqrt{\frac{p}{2\pi m\hbar}}\mathrm{e}^{-\mathrm{i}p^2t/2m\hbar}\psi(p)\right|^2. \tag{5.10}$$

For a more general arrival point $x=x_A$, Kijowski's distribution can be written as

$$\begin{aligned}\Pi_K(x_A,t)&=\frac{\hbar}{2\pi m}\left|\int_0^\infty \mathrm{d}k\widetilde{\psi}(k)\sqrt{k}\mathrm{e}^{-\mathrm{i}\hbar k^2t/2m}\mathrm{e}^{\mathrm{i}kx_A}\right|^2 &(5.11)\\ &=\langle\psi_t|\frac{1}{m}\widehat{p}^{1/2}\delta(x-x_A)\widehat{p}^{1/2}|\psi_t\rangle, &(5.12)\end{aligned}$$

where $\widetilde{\psi}(k)\equiv\psi(p=k\hbar)$. The operator $\widehat{p}^{1/2}\delta(x-x_A)\widehat{p}^{1/2}$ can be regarded as the symmetrized quantization of the phase-space function (2.3). Since only the free-motion case is involved, ψ_t has the relation with the initial state $\psi=\psi_{t=0}$:

$$\psi_t(p)=\mathrm{e}^{-\mathrm{i}p^2t/2m\hbar}\psi(p). \tag{5.13}$$

For positive momentum components, Kijowski's distribution is equivalent to that has been obtained using positive operator valued measures (POVMs). Werner pointed out a basic property of the distribution, the covariance under time translations, is given by

$$\Pi_K(x_A,t;\psi(t_0))=\Pi_K(x_A,t+t_0;\psi(0)). \tag{5.14}$$

By Fourier transformation, the current density and Kijowski's distribution can be expressed in the same form in momentum space in terms of the kernel functions $f_J(k,k')=(k+k')/2$ and $f_K(k,k')=\sqrt{kk'}$ respectively:

$$J(x_A,t)=\frac{\hbar}{2\pi m}\int_0^\infty \mathrm{d}k\mathrm{d}k'\overline{\widetilde{\psi}(k)}\widetilde{\psi}(k')\mathrm{e}^{\mathrm{i}\hbar(k^2-k'^2)t/2m}\mathrm{e}^{-\mathrm{i}(k-k')x_A}f_J(k,k') \tag{5.15}$$

$$\Pi_K(x_A,t)=\frac{\hbar}{2\pi m}\int_0^\infty \mathrm{d}k\mathrm{d}k'\overline{\widetilde{\psi}(k)}\widetilde{\psi}(k')\mathrm{e}^{\mathrm{i}\hbar(k^2-k'^2)t/2m}\mathrm{e}^{-\mathrm{i}(k-k')x_A}f_K(k,k') \tag{5.16}$$

The average of t under Kijowski's distribution Π_K coincides with that computed with the probability flux $J(t)$:

$$\begin{aligned}\langle t\rangle &= \frac{m}{\hbar}\int_0^\infty \mathrm{d}k|\widetilde{\psi}(k)|^2\frac{1}{k}\left(x_A+\frac{\partial}{\partial k}\arg\widetilde{\psi}(k)\right) \\ &= \int_{-\infty}^{\infty}\mathrm{d}tJ(x_A,t)=\int_{-\infty}^{\infty}\mathrm{d}t\Pi_K(x_A,t), \end{aligned}\tag{5.17}$$

where $\widetilde{\psi}(k)=|\widetilde{\psi}(k)|\exp(\mathrm{i}\arg\widetilde{\psi}(k))$. In the case of Gaussian wave packets, we have the expression of $\widetilde{\psi}(k)$ as follows:

$$\widetilde{\psi}(k)=\frac{1}{\sqrt{\Delta k\sqrt{2\pi}}}\mathrm{e}^{-(k-k_0)^2/(4\Delta k^2)}\mathrm{e}^{-\mathrm{i}kx_0}, \tag{5.18}$$

where x_0 is the mean position and k_0 the mean momentum with Δk as the momentum spread at time $t=0$. Simple calculation on Eq. (5.18) immediately yields that

$$\frac{\partial}{\partial k}\arg\widetilde{\psi}(k)=-x_0 \tag{5.19}$$

and hence the mean arrival time of the quantum version of Eq. (2.1),

$$\langle t\rangle=\frac{m}{\hbar}\int_0^\infty \mathrm{d}k|\widetilde{\psi}(k)|^2\frac{1}{k}(x_A-x_0). \tag{5.20}$$

Kijowski argued that the expression for arrivals from the right side are analogous to that from the left, which is represented by the above $\Pi_K(t)$. Therefore the total probability density for arrivals at time t at position x_A of a freely one-dimensional particle is given by

$$\Pi_K(x_A,t)=\frac{\hbar}{2\pi m}\sum_{\pm}\left|\int_0^\infty \mathrm{d}k\widetilde{\psi}(\pm k)\sqrt{k}\mathrm{e}^{-\mathrm{i}\hbar k^2t/2m}\mathrm{e}^{\pm\mathrm{i}kx_A}\right|^2. \tag{5.21}$$

Kijowski claimed that an operator that possesses the characteristics required for an arrival-time operator had been found due to the bilinear functional. Unfortunately he also proved that the operator fails to have a self-adjoint extension. Moreover, the generalization of Kijowski's distribution for particles in external potentials is still subject to debate[32][33].

6 Aharonov-Bohm arrival-time operator and POVMs

In an important paper[13], Aharonov and Bohm considered the free motion as a clock to measure time using the position and momentum of a free particle. The corresponding time operator is defined by

$$\widehat{T}_{AB} := -\frac{m}{2}\left((\widehat{x}-x_A)\widehat{p}^{-1}+\widehat{p}^{-1}(\widehat{x}-x_A)\right). \tag{6.1}$$

Despite that there is a minus sign in the definition formula compared to Eq. (3.1) and thus implies that $\mathrm{d}\widehat{T}_{AB}(t)/\mathrm{d}t=-1$ in Heisenberg picture, we still refer to this operator as the Aharonov-Bohm arrival-time operator. It is yet not, and, according to Pauli's argument, cannot be a self-adjoint arrival-time operator on a dense domain, nevertheless it is maximally symmetric and fulfils the desired commutation relation with the free one-dimensional Hamiltonian, $\widehat{H}_0=\widehat{p}^2/2m$, i.e., $[\widehat{H}_0,\widehat{T}_{AB}]=\mathrm{i}\hbar$. It is therefore a good candidate for the time operator.

Since this time operator is expressed in terms of $\widehat{p}^{-1}$, it is convenient to consider the Hilbert space in the momentum representation. We then have

$$\widehat{T}_{AB}(p)=\frac{\mathrm{i}\hbar m}{2}\left(\frac{1}{p^2}-\frac{2}{p}\frac{\partial}{\partial p}\right). \tag{6.2}$$

Alternatively, we can write it as

$$\widehat{T}_{AB}(p)=-\mathrm{i}\hbar m\frac{1}{p^{1/2}}\frac{\partial}{\partial p}\frac{1}{p^{1/2}}, \tag{6.3}$$

which will be valid for $p\neq 0$.

To avoid the singularity we clearly find that the domain of the differential operator $\widehat{T}_{AB}$ with expression (6.2) should be the subset of the set $\{\psi:\psi(p)\sim p^{1/2}$, or $\psi(p)p^{-3/2}\to 0, \text{as } p\to 0\}$. Also, ψ are required to be absolutely continuous square integrable functions due to the fact that $\widehat{T}_{AB}\psi$ belongs to the Hilbert space in the momentum representation. Demanding, out of the requirement for $\widehat{T}_{AB}$ to be symmetric, that $\langle\widehat{T}_{AB}\psi|\psi\rangle=\langle\psi|\widehat{T}_{AB}\psi\rangle$ for any ψ in the domain of $\widehat{T}_{AB}$, we eventually obtain by integration by parts that the first possibility, i.e., $\psi(p) \sim p^{1/2}$, as $p \to 0$, should be neglected and thus the domain of the time operator is the set of absolutely continuous square integrable functions ψ of p such that $\psi(p)p^{-3/2} \to 0$, as $p \to 0$ and $\|\widehat{T}_{AB}\psi\|^2$ is finite.

As for the alternative expression (6.3), letting $(\Theta(p) - \mathrm{i}\Theta(p))\psi(p)/\sqrt{|p|}$ be absolutely continuous, we see that, under such a condition, its domain coincides with that of the expression (6.2) and therefore both expressions are equivalent to each other.

It can be found that there exist two independent eigenvectors of $\widehat{T}_{AB}^{\dagger}$ with eigenvalue i and none with eigenvalue $-i$. The weak eigenvectors of $\widehat{T}_{AB}$ are then given by

$$\psi_{\pm}(p) = \Theta(\pm p)\ \sqrt{\pm p}\ \exp(-p^2/2m\hbar). \tag{6.4}$$

The deficiency indices for this case are $(2,0)$ and $\widehat{T}_{AB}$ is a maximally symmetric operator, which implies there does not exist any self-adjoint extension [34]. It is convenient, from the mathematical standpoint, to change to the energy representation, which corresponds to the decomposition of $\widehat{T}_{AB}$ with deficiency indices $(2,0)$ into two operators with indices $(1,0)$,

$$\widehat{T}_{AB} = \widehat{T}_{+} \oplus \widehat{T}_{-}, \tag{6.5}$$

where $\widehat{T}_{+}$ and $\widehat{T}_{-}$ are both isomorphic to the momentum operator on the half-line. And correspondingly, the Hilbert space $\mathcal{H}_p$ is in the meanwhile decomposed into the subspace of positive momentum and that of negative momentum,

$$\begin{aligned} L^2(\mathbb{R}, \mathrm{d}p) &= L^2(\mathbb{R}^+, \mathrm{d}E) \oplus L^2(\mathbb{R}^-, \mathrm{d}E) \\ &= \mathcal{H}_+ \oplus \mathcal{H}_-, \end{aligned}$$

where $\mathcal{H}_+$ corresponds to positive p and $\mathcal{H}_-$ negative p. For any $\psi \in \mathcal{H}_p$, the decomposition reads,

$$\psi(p) = \left(\frac{|p|}{m}\right)^{1/2} \left[\Theta(p)\psi_+\left(\frac{p^2}{2m}\right) + \Theta(-p)\psi_-\left(\frac{p^2}{2m}\right)\right], \tag{6.6}$$

where $\psi_\pm \in \mathcal{H}_\pm$ are given by,

$$\psi_\pm(E) = (m/2E)^{1/4}\psi(\pm\sqrt{2mE}). \tag{6.7}$$

Under this isomorphism, the decomposition of Aharonov-Bohm time operator $\widehat{T}_{AB}$ (6.5) is given by

$$\widehat{T}_{AB} = (-\mathrm{i}\hbar\partial_E) \oplus (-\mathrm{i}\hbar\partial_E), \tag{6.8}$$

and correspondingly the generalized eigenfunctions ψ_p are transformed into the forms in terms of the parameters t and $\pm$:

$$\psi_+^{(t)}(E) = \left(\frac{1}{\sqrt{2\pi\hbar}}\exp(\mathrm{i}Et/\hbar),\ 0\right) \tag{6.9}$$

and

$$\psi_-^{(t)}(E) = \left(0,\ \frac{1}{\sqrt{2\pi\hbar}}\exp(\mathrm{i}Et/\hbar)\right), \tag{6.10}$$

In momentum representation, $\psi_\alpha^{(t)}(E)$ are transformed into the following forms:

$$\tilde{\psi}_\alpha^{(t)}(p) = \Theta(\alpha p)\left(\frac{\alpha p}{2\pi m\hbar}\right)^{1/2}\exp(\mathrm{i}p^2t/2m\hbar). \tag{6.11}$$

Introduce Dirac's notation and define $|t,\alpha\rangle$ by

$$\langle p|t,\alpha\rangle = \tilde{\psi}_\alpha^{(t)}(p). \tag{6.12}$$

It is straightforward to prove that the states $|t,\pm\rangle$ are not orthogonal:

$$\langle t',\alpha'|t,\alpha\rangle = \frac{\delta_{\alpha,\alpha'}}{2}\left(\delta(t-t') + \frac{\mathrm{i}}{\pi}\right), \quad \alpha,\, \alpha' = \pm. \tag{6.13}$$

But they are complete, namely,

$$\sum_\alpha \int_{-\infty}^{\infty} \mathrm{d}t|t,\alpha\rangle\langle t,\alpha| = \widehat{1}. \tag{6.14}$$

This behooves us to define an arrival-time distribution in terms of a positive operator valued measure (POVM).

Each observable is related to a positive operator valued measure (POVM). The POVM is a mapping from intervals σ of the real axis to positive operators $\widehat{A}_\sigma$ which are additive with respect to disjoint intervals σ, namely,

$$\widehat{A}_{\sigma\cup\sigma'} = \widehat{A}_\sigma + \widehat{A}_{\sigma'}, \quad \text{if } \sigma\cap\sigma' = \emptyset, \tag{6.15}$$

and which add up to unity when the whole set of intervals σ are summed,

$$\sum_\sigma \widehat{A}_\sigma = \widehat{1}. \tag{6.16}$$

POVMs lead to a generalization of quantum mechanics, where each observable is linked to a certain corresponding self-adjoint operator and the probability is obtained by calculating the expectation value of the corresponding projection operator. The POVM theory indicates that it is sufficient to derive the measurement probabilities with the eigenstates of some maximally symmetric operator.

The POVM of arrival times is defined in terms of the eigenstates (6.4),

$$\widehat{\Pi}(t_2,t_1) = \sum_\alpha \int_{t_1}^{t_2} \mathrm{d}t |t,\alpha\rangle\langle t,\alpha|, \tag{6.17}$$

and the probability distribution for arrival times is therefore given by

$$\widehat{\Pi}_{\mathrm{POVM}}(t) = \langle\psi|\left(\sum_\pm |t,\pm\rangle\langle t,\pm|\right)|\psi\rangle = \sum_\pm |\langle t,\pm|\psi\rangle \tag{6.18}$$

$$= \frac{\hbar}{2\pi m}\sum_\pm \left|\int_0^\infty \mathrm{d}k\widetilde{\psi}(\pm k)\sqrt{k}\mathrm{e}^{-\mathrm{i}\hbar k^2 t/2m}\mathrm{e}^{\pm\mathrm{i}kx_A}\right|^2, \tag{6.19}$$

which coincides with the total Kijowski's distribution (5.21). In contrast to the quantum mechanical flux, the distribution obtained in this way is positive.

7 Arrival-time distributions and the relation to local densities

Arrival-time distributions are, in essence, quantum local densities associated with the quantum flux. For a quantum observable not diagonal in position representation, we may obtain the expression of the local density starting from its classical counterpart. The classical local density $\alpha_A(x,t)$ corresponding to the classical dynamical variable $A(y,p)$ can be expressed as,

$$\alpha_A(x,t) = \int \mathrm{d}y \int \mathrm{d}p \ \varrho(y,p;t)A(y,p)\delta(y-x), \tag{7.1}$$

where $\varrho(y,p;t)$ is the phase-space density. It is easy to see that we may obtain the average value of $A(y,p)$ by integrating the local density $\alpha_A(x,t)$ over x,

$$\int \mathrm{d}x \ \alpha_A(x,t) = \int \mathrm{d}x \int \mathrm{d}p \ \varrho(x,p;t)A(x,p) = \langle A\rangle_t. \tag{7.2}$$

To obtain the quantization of the local density, it is of importance and convenience to begin from quantizing A, i.e., seeking a proper operator expression $\widehat{A}(x)$ for A such that the expectation value of $\widehat{A}(x)$ with respect to a certain quantum state ψ_t give the quantum local density value at x, namely,

$$\alpha_A(x,t) = \langle\psi_t|\widehat{A}(x)|\psi_t\rangle. \tag{7.3}$$

Notice that the delta function corresponds to the operator identity

$$\delta(\widehat{x}-x) = |x\rangle\langle x|, \tag{7.4}$$

we may consider the operator

$$\widehat{A}(x) = \widehat{A}|x\rangle\langle x| \tag{7.5}$$

as a quantum density operator for the variable A. However, in the case that $\widehat{A}$ is not diagonal in position representation, $\widehat{A}$ does not commute with $|x\rangle\langle x|$ and therefore we should choose the symmetric forms for $\widehat{A}$. There are many symmetrization of their product, such as

$$\widehat{A}(x) = \widehat{A}^{1/2}|x\rangle\langle x|\widehat{A}^{1/2}, \tag{7.6}$$

$$\text{or} \quad \widehat{A}(x) = \frac{1}{2}\left(\widehat{A}|x\rangle\langle x| + |x\rangle\langle x|\widehat{A}\right), \tag{7.7}$$

$$\text{or} \quad \widehat{A}(x) = \frac{1}{2}\widehat{A}^{1/2}|x\rangle\langle x|\widehat{A}^{1/2} + \frac{1}{4}\left(\widehat{A}|x\rangle\langle x| + |x\rangle\langle x|\widehat{A}\right), \tag{7.8}$$

and hence many local quantum densities $\alpha_A(x,t)$. It is useful to notice that the first symmetrization form indicates a positive quantum density while the others may provide negative values if we choose certain values of x and t.

Even if $\widehat{A}$ and $|x\rangle\langle x|$ do not commute with each other, it does not necessarily lead to the consequence that there is only one symmetrization that represents the true operator. Different symmetrized forms may imply different latent properties that may be revealed through different experimental measurement approaches. Actually only by taking experimental measurement procedures can we judge which distribution is physically meaningful.

Take velocity density for example. The symmetrization of the quantum operators for velocity density Eq. (2.3) can be

$$\widehat{J}_1(x) = \frac{1}{m}\widehat{p}^{1/2}|x\rangle\langle x|\widehat{p}^{1/2}, \tag{7.9}$$

$$\text{or}\quad \widehat{J}_2(x) = \frac{1}{2m}\left(\widehat{p}|x\rangle\langle x| + |x\rangle\langle x|\widehat{p}\right), \tag{7.10}$$

or etc.. Taking the expectation values of the two operators with respect to ψ_t, we arrive at Kijowski's distribution $\Pi_K(x,t)$ and the quantum mechanical current density $J(x,t)$ separately:

$$\Pi_K(x,t) = \frac{1}{m}\langle\psi_t|\widehat{p}^{1/2}|x\rangle\langle x|\widehat{p}^{1/2}|\psi_t\rangle, \tag{7.11}$$

$$J(x,t) = \frac{1}{2m}\langle\psi_t|\left(\widehat{p}|x\rangle\langle x| + |x\rangle\langle x|\widehat{p}\right)|\psi_t\rangle. \tag{7.12}$$

For the quantum mechanical flux $J(x,t)$, the quantization rule is called Weyl-Wigner rule[36]. As mentioned above, the first symmetrized operator for the velocity density, which yields Kijowski's distribution $\Pi_K(x,t)$, is positive while the second one, which leads to the flux $J(x,t)$, may become negative and display the backflow effect.

8 Arrival-time distribution in small complex potentials

The notion of complex potentials was first introduced into arrival time problem by Allcock[10] in his attempt to construct a specific detector model. He chose a pure imaginary potential step

$$V(x) = -iV_0\Theta(x) \tag{8.1}$$

and found that the particle will not be absorbed, i.e., detected, but undergo considerable reflection, as V_0 becomes very large (known later as the Zeno limit), whereas in the opposite case when V_0 tends to 0, the particle will be absorbed.

Here we give an introduction to a model with this complex potential considered by J. Halliwell and J. Yearsley[54], which involves the calculation of the arrival-time distribution and illustrates the connection between the distribution and the current density. These authors start from the case of small V_0 in Eq. (8.1) and the particle of negative momenta with its initial wave function concentrated in the half line $x > 0$.

The evolution of the quantum state for the particle at interval τ reads

$$\begin{aligned} |\psi(\tau)\rangle &= \exp(-iH\tau/\hbar)|\psi\rangle \\ &= \exp(-iH_0\tau/\hbar - V_0\Theta(-x)\tau/\hbar)|\psi\rangle, \end{aligned} \tag{8.2}$$

where H_0 is the free Hamiltonian and $H = H_0 - iV_0\Theta(-x)$. Since

$$N(\tau) = \langle\psi(\tau)|\psi(\tau)\rangle \tag{8.3}$$

is the part of the wave packet that remains unabsorbed by time τ, i.e., the probability of having not crossed $x = 0$ during $[0, \tau]$, then the probability flux of crossing during $[\tau, \tau + \mathrm{d}\tau]$ is

$$\Pi(\tau) = -\frac{\mathrm{d}N}{\mathrm{d}\tau}. \tag{8.4}$$

The distribution for the case of $V_0 \to 0$ without significant reflection is proposed first by Allcock[10] and recently by some other authors[55][56] to have such a form,

$$\Pi(\tau) = \int_{-\infty}^{\infty} \mathrm{d}t\, R(V_0, \tau - t)J(t), \tag{8.5}$$

where $J(t)$ is the current density,

$$J(\tau) := -\frac{\hbar}{2m}\langle\psi_{\rm f}(\tau)|[\widehat{p}\delta(\widehat{x}) + \delta(\widehat{x})\widehat{p}]|\psi_{\rm f}(\tau)\rangle, \tag{8.6}$$

with $\psi_{\rm f}$ being the freely evolved state, and $R(V_0, t)$ is the apparatus resolution function,

$$R(V_0, t) = 2V_0\Theta(t)\exp(-2V_0 t). \tag{8.7}$$

Simple calculation for (8.4) using Eq. (8.2) and Eq. (8.3) yields

$$\begin{aligned}\Pi(\tau) &= -\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\psi_\tau|\psi_\tau\rangle \\ &= -\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\psi|\exp\{-2V_0\Theta(-\widehat{x})\tau\}|\psi\rangle \\ &= 2V_0\langle\psi_\tau|\Theta(-\widehat{x})|\psi_\tau\rangle.\end{aligned} \tag{8.8}$$

One way to calculate the arrival-time distribution given by Eq. (8.8) is to use path decomposition expansion (PDX). For any propagator of such a form

$$g(x_1, \tau; x_0, 0) = \langle x_1|\exp(-iH_0\tau/\hbar - V_0\Theta(-\widehat{x})f(\widehat{x})\tau/\hbar|x_0\rangle, \quad x_1,\, x_0 > 0, \tag{8.9}$$

One may employ the path integral technique and sum over all possible paths between $x(0) = x_0$ and $x(\tau) = x_1$, i.e.,

$$g(x_1, \tau; x_0, 0) = \int \mathcal{D}x \exp(iS/\hbar), \tag{8.10}$$

where

$$S = \int_0^\tau \mathrm{d}t\left(\frac{1}{2}m\dot{x}^2 + iV_0\Theta(-x)f(x)\right), \tag{8.11}$$

to obtain the result,

$$g(x_1, \tau; x_0, 0) = \frac{i}{2m}\int_0^\tau \mathrm{d}t_1\, g(x_1, \tau; 0, t_1)\frac{\partial g_r}{\partial x}(x, t_1; x_0, 0)|_{x=0}, \tag{8.12}$$

where $g_r(x, t; x_0, 0)$ is the restricted propagator derived by summing over all paths restricted to $x(t) > 0$.

Here one may rewrite Eq. (8.12) as the operator form,

$$\langle x|\exp(-iH\tau|\psi\rangle = -\frac{1}{m}\int_0^\tau \mathrm{d}t\langle x|\exp(-iH(\tau-t))\delta(\widehat{x})\widehat{p}\exp(-iH_0 t)|\psi\rangle. \tag{8.13}$$

Using $1 = \int \mathrm{d}x|x\rangle\langle x|$ and $\delta(\widehat{x}) = |0\rangle\langle 0|$, inserting Eq. (8.13) in Eq. (8.8) and changing the variables $s = \tau - t$, $s' = \tau - t'$, one obtain

$$\begin{aligned}\Pi(\tau) = {} & \frac{2V_0}{m^2}\int_0^\tau \mathrm{d}s' \int_0^\tau \mathrm{d}s \int_{-\infty}^0 \mathrm{d}x\,\langle 0|\exp(iH^\dagger s')|x\rangle\langle x|\exp(-iHs)|0\rangle \\ & \times\,\langle\psi|\exp(iH_0(\tau-s'))\,\widehat{p}\,\delta(\widehat{x})\,\widehat{p}\,\exp(-iH_0(\tau-s))|\psi\rangle.\end{aligned} \tag{8.14}$$

For the case where $V_0 \to 0$, the integral of the propagator will mainly be attributable to those paths in the neighbourhood of the straight path from $x = 0$ to $x < 0$, therefore one may approximate the propagator as

$$\begin{aligned}\langle x|\exp(-iHs)|0\rangle &\approx \langle x|\exp(-iH_0 s)|0\rangle\exp(-V_0 s) \\ &\approx \left(\frac{m}{2\pi i s}\right)^{1/2}\exp\left(i\frac{mx^2}{2s} - V_0 s\right).\end{aligned} \tag{8.15}$$

Integration of x by means of this semiclassical approximation yields

$$\begin{aligned}\Pi(\tau) = {} & \frac{V_0}{m^2}\int_0^\tau \mathrm{d}s' \int_0^\tau \mathrm{d}s \left(\frac{m}{2\pi i}\right)^{1/2}\frac{\mathrm{e}^{-V_0(s+s')}}{(s-s')^{1/2}} \\ & \times\,\langle\psi_\tau|\exp(-iH_0 s')\,\widehat{p}\,\delta(\widehat{x})\,\widehat{p}\,\exp(iH_0 s)|\psi_\tau\rangle.\end{aligned} \tag{8.16}$$

Using

$$\int_0^\tau \mathrm{d}s' \int_0^\tau \mathrm{d}s = \int_0^\tau \mathrm{d}s' \int_{s'}^\tau \mathrm{d}s + \int_0^\tau \mathrm{d}s \int_s^\tau \mathrm{d}s' \tag{8.17}$$

and changing the variables, $u = s'$, $v = s - s'$ in the first integral, and $u = s$, $v = s' - s$ in the second integral, one has

$$\begin{aligned}\Pi(\tau) = {} & \frac{V_0}{m^2}\left(\frac{m}{2\pi}\right)^{1/2}\int_0^\tau \mathrm{d}u\mathrm{e}^{-2V_0 u}\int_0^{\tau-u}\mathrm{d}v\frac{\mathrm{e}^{-V_0 u}}{v^{1/2}} \\ & \times\left[\frac{1}{i^{1/2}}\langle\psi_\tau|\exp(-iH_0 u)\,\widehat{p}\,\delta(\widehat{x})\,\widehat{p}\,\exp(iH_0(u+v))|\psi_\tau\rangle\right. \\ & \left. +\ \frac{1}{(-i)^{1/2}}\langle\psi_\tau|\exp(-iH_0(u+v))\,\widehat{p}\,\delta(\widehat{x})\,\widehat{p}\,\exp(iH_0 u)|\psi_\tau\rangle\right].\end{aligned} \tag{8.18}$$

Taking the assumption that $V_0\tau \gg 1$, one obtains

$$\begin{aligned}
\Pi(\tau) &= 2V_0 \int_0^\tau \mathrm{d}u\, \mathrm{e}^{-2V_0 u} \cdot \frac{1}{2m} \langle \psi_{\tau-u} | \widehat{p}\, \delta(\widehat{x})\, \Sigma(\widehat{p}) + \Sigma^\dagger(\widehat{p})\, \delta(\widehat{x})\, \widehat{p} | \psi_{\tau-u} \rangle \quad (8.19)\\
&\approx 2V_0 \int_0^\tau \mathrm{d}u\, \mathrm{e}^{-2V_0 u} \cdot \frac{1}{2m} \langle \psi_{\tau-u} | \widehat{p}\, \delta(\widehat{x})\, \mathrm{sgn}(\widehat{p}) + \mathrm{sgn}^\dagger(\widehat{p})\, \delta(\widehat{x})\, \widehat{p} | \psi_{\tau-u} \rangle \\
&= 2V_0 \int_0^\tau \mathrm{d}t\, \mathrm{e}^{-2V_0(\tau-t)} \cdot \frac{-1}{2m} \langle \psi_t | \widehat{p}\, \delta(\widehat{x}) + \delta(\widehat{x})\, \widehat{p} | \psi_t \rangle \\
&\equiv 2V_0 \int_0^\tau \mathrm{d}t\, \mathrm{e}^{-2V_0(\tau-t)} \cdot J(t). \qquad (8.20)
\end{aligned}$$

where

$$\Sigma(\widehat{p}) := \frac{\widehat{p}}{[2m(H_0 + iV_0)]^{1/2}} \qquad (8.21)$$

and taking the limit $V_0 \to 0$, $\Sigma(\widehat{p})$ can be approximated as $\mathrm{sgn}(\widehat{p})$, the sign function and thus equals to -1 since one considers the particle of negative momentum, and

$$J(t) = \frac{-1}{2m} \langle \psi_t | \widehat{p}\, \delta(\widehat{x}) + \delta(\widehat{x})\, \widehat{p} | \psi_t \rangle \qquad (8.22)$$

is exactly the current density (7.12) setting $x = 0$, wherein the minus sign is due to the condition that the particle starts from the right half line with negative momentum.

Note that (8.14) is positive, one can see that Eq. (8.19), directly derived from it, preserves the positivity since the semiclassical approximation and the assumption $V_0\tau \gg 1$ will not violate the positivity. Eq. (8.20) is, however, not necessarily positive according to the thorough discussion by J. Halliwell and J. Yearsley[54], which is in agreement with the fact that J is not always positive due to the existence of backflow effect.

9 Phase times, tunneling times and transmission times

The presence of interactions greatly enhances the difficulties and puzzling aspects of arrival times. The famous question ”How long does it take a particle to traverse a potential barrier?” has caused a long controversy in the past with many authors engaged in this area, holding rather different views and claiming their own proposal for tunneling or traversal time to be the good one. This debate, however, is moving towards a public agreement that there is no single quantity that includes the whole and only truth about timing the particle's traversal, see e.g.[37]. As the example of velocity density, which has various quantization formulations such as Eq. (7.9) and Eq. (7.10), the tunneling time is in essence also an example of quantization for a classical quantity that involves products of noncommuting observables: a unique, classical question corresponds to different quantum versions.

In 1955, Wigner considered the peaks of incoming waves as well as transmitted wave packets by means of stationary phase approximation and introduced the concept of phase times, which has the following expression in the tunneling case,

$$t_{\mathrm{ph}}(k) = \frac{m}{\hbar k}\left(L + \frac{\partial}{\partial k}\arg T(k)\right), \tag{9.1}$$

where $L = x_2 - x_1$ is the width of potential region and $T(k)$ the tunneling amplitude. For monochromatic waves, $mL/\hbar k$ in this equation is the time it takes the free packet to traverse the potential region, hence $m\frac{\partial}{\partial k}(\arg T(k))/\hbar k$ is the time delay concerned with the transmission. Phase times have laid the foundations of investigating into the Hartman effect for potential barriers [39][40][41][42].

Since the notion of phase times was introduced based on the stationary phase approximation and asymptotic approaches, it is to some extent a source of confusion and not a clearly defined concept. Moreover, the peak is not a reliable property of the wave packet in that tunneling is possible to distort the wave packet and therefore an incoming peak does not necessarily turn into an outgoing peak[43].

In 1992, transmission times τ_T and reflection times τ_R were introduced via the average probability flux of transmitted and reflection respectively as well as the average incoming flux[44][45]. The wave packet passes through x_1 before t_c and after some duration of time the reflected part goes back from the point x_2 to the point x_1 as a negative current while the transmitted part passes through x_2. The expression of τ_T and τ_R are given by

$$\tau_{\mathrm{T}} = \langle t \rangle_{x_2}^{\mathrm{out}} - \langle t \rangle_{x_1}^{\mathrm{in}}, \tag{9.2}$$

$$\tau_{\mathrm{R}} = \langle t \rangle_{x_1}^{\mathrm{out}} - \langle t \rangle_{x_1}^{\mathrm{in}}, \tag{9.3}$$

where

$$\langle t \rangle_{x_2}^{\mathrm{out}} = \frac{\int_{-\infty}^{\infty} \mathrm{d}t \ tJ(x_2, t)}{\int_{-\infty}^{\infty} \mathrm{d}t \ J(x_2, t)}, \tag{9.4}$$

$$\langle t \rangle_{x_1}^{\mathrm{in}} = \int_{-\infty}^{t_c} \mathrm{d}t \ tJ(x_1, t), \tag{9.5}$$

$$\langle t \rangle_{x_1}^{\mathrm{out}} = -\frac{\int_{t_c}^{\infty} \mathrm{d}t \ tJ(x_1, t)}{\int_{t_c}^{\infty} \mathrm{d}t \ |J(x_1, t)|}. \tag{9.6}$$

Transmission and reflection times are usually linked to the dwell time τ_{D}, i.e., the time for which a particle stays in a given region. The relation between the two times reads

$$\tau_{\mathrm{D}}(k) = |T(k)|^2 \tau_{\mathrm{T}} + |R(k)|^2 \tau_{\mathrm{R}}, \tag{9.7}$$

where $T(k)$ and $R(k)$ are the transmission and reflection amplitudes of the plane wave solutions for the given potential region, separately.

10 Some interpretations of time-energy uncertainty relation

The Heisenberg uncertainty relation is expressed in terms of the quantum operators for position, $\widehat{x}$ and for momentum, $\widehat{p}$. One formal version for the uncertainty relation was given by,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}, \tag{10.1}$$

where σ_x and σ_p are the standard deviations of the two operators $\widehat{x}$ and $\widehat{p}$, respectively. This expression was derived first by Kennard[46] and later by

Weyl[47]. Robertson[48] then went further to present a generalized version of equation (10.1) for arbitrary Hermitian operators $\widehat{A}$ and $\widehat{B}$

$$\sigma_A \sigma_B \geq \frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle. \tag{10.2}$$

Derivation of the Robertson uncertainty relation runs as follows. Let

$$|f\rangle = |(A - \langle A\rangle)\Psi\rangle \tag{10.3}$$

and

$$|g\rangle = |(B - \langle B\rangle)\Psi\rangle, \tag{10.4}$$

then based on the definition of standard deviation, we have,

$$\sigma_A^2 = \langle(A - \langle A\rangle)\Psi|(A - \langle A\rangle)\Psi\rangle = \langle f|f\rangle \tag{10.5}$$

and

$$\sigma_B^2 = \langle(B - \langle B\rangle)\Psi|(B - \langle B\rangle)\Psi\rangle = \langle g|g\rangle. \tag{10.6}$$

Applying the Schwarz inequality,

$$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2, \tag{10.7}$$

and the inequality associated with squared modulus,

$$|z|^2 \geq \Big(\mathrm{Im}(z)\Big)^2 = \Big(\frac{z - z^*}{2i}\Big)^2, \tag{10.8}$$

we obtain the inequality,

$$\sigma_A^2\sigma_B^2 \geq |\langle f|g\rangle|^2 \geq \Big(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\Big)^2. \tag{10.9}$$

Notice that

$$\begin{aligned}\langle f|g\rangle &= \langle\Psi|(A - \langle A\rangle)(B - \langle B\rangle)\Psi\rangle \\ &= \langle AB\rangle - \langle A\rangle\langle B\rangle\end{aligned} \tag{10.10}$$

and similarly,

$$\langle g|f\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle, \tag{10.11}$$

we have

$$\langle f|g\rangle - \langle g|f\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle - \langle BA\rangle + \langle A\rangle\langle B\rangle = \langle [A, B]\rangle. \tag{10.12}$$

Substituting Eq. (10.12) into the inequality (10.9) yields

$$|\langle f|g\rangle|^2 \geq \left(\frac{1}{2i}\langle [A, B]\rangle\right)^2. \tag{10.13}$$

which immediately implies that

$$\sigma_A \sigma_B \geq \frac{1}{2i}\langle [A, B]\rangle. \tag{10.14}$$

The general Robertson uncertainty relation holds perfectly for position and momentum–to see this clearly, just substitute the two operators for position and momentum into Eq. (10.14). However, since there is, in universal sense, no self-adjoint time operator, the uncertainty relation between time and energy is not a straightforward consequence of the general Robertson uncertainty relation for noncommuting operators. Note that time bears the same complementary relation to energy as position does to momentum according to relativity, it was strongly suggested by many authors that there should be such a inequality as follows:

$$\Delta E \Delta t \gtrsim h. \tag{10.15}$$

However the central problem lies in what Δt exactly means. As is mentioned in the beginning of this paper, time can be viewed as an external parameter that does not quite clearly belong to a particle or a given state. While the position in the position-momentum uncertainty relation is referred to as where a particle is, or to be more precisely, has a certain probability to be and is independent of where the measuring apparatus is operated, the time in the uncertainty relation is easily understood as exactly equal to, and has close relation to when the experiment apparatus is turned on or off and can therefore be known precisely regardless of what property the apparatus is measuring. In this sense, it seems that time and energy can be determined simultaneously, as L. Landau stated once "To violate the time-energy uncertainty relation all I have to do is measure the energy very precisely and then look at my watch!" [49]

Another formulation of the energy-time uncertainty relation argues that to measure the energy of a given quantum state to an accuracy ΔE requires a period $\Delta t > h/\Delta E$. Here Δt in the time-energy uncertainty relation (10.15) is identified not as an uncertainty but as the external time duration within which the measurement of energy is completed. In the meanwhile, there are two ways of interpreting ΔE: one is to regard ΔE as the range by which the measurement changes the energy of the object due to the inevitable interaction exerted by the measurement; another is to consider ΔE as the resolution of measuring energy.

This formulation bears some resemblance to the former one with respect to the notion of time as an external parameter. However, in 1961 Y. Aharonov and D. Bohm[13] invalidated the formulation and by constructing an energy measurement model they presented a counter example demonstrating that the external time-energy uncertainty relation is not universally valid–it may be valid in some case, but may be wrong for some types of Hamiltonians. This model starts with a system of two one-dimensional particles, of which one particle is the object and the other serves as a probe to measure the momentum of the former one. The total Hamiltonian then reads

$$H = \frac{P_x^2}{2M_x} + \frac{P_y^2}{2M_y} + YP_x g(t), \tag{10.16}$$

where X and Y are the positions of the object and the probe respectively, P_x and P_y are their momenta, and M_x and M_y their masses. The interaction term $YP_xg(t)$ represents a coupling effect between both momenta P_x and P_y. And $g(t)$ provides the strength of the interaction in and only in a specified time interval:

$$g(t) = \begin{cases} g_0 \text{ if } \ 0 \leq t \leq \Delta t, \\ 0 \ \ \text{otherwise.} \end{cases}$$

The motion equations for the positions and momenta in Heisenberg picture are given by,

$$\dot{X} = \tfrac{1}{M_x}P_x + Yg(t),\ \dot{P}_x = 0,$$
$$\dot{Y} = \tfrac{1}{M_y}P_y, \qquad\qquad \dot{P}_y = -P_x g(t).$$

Solving the equations, we have

$$P_x = P_x^0, \quad P_y = P_y^0 - P_x^0 g_0 \Delta t, \quad \text{for } t \geq \Delta t. \tag{10.17}$$

The operator for the kinetic energy of the object is given by

$$H_0 = \frac{M_x}{2}\dot{X}^2 = \frac{P_x^2}{2M_x}. \tag{10.18}$$

It follows that before and after the measurement the momentum P_x and hence the kinetic energy of the object are the same, although they vary during the interaction. This implies that a reproducibility of energy measurement is possible at least for some types of Hamiltonians. From (10.17) we may obtain the following condition:

$$\Delta P_x g_0 \Delta t \cong \Delta P_y^0, \tag{10.19}$$

which means it is possible to make Δt and ΔP_x (and hence the resolution ΔH_0) arbitrarily small by giving an enough large g_0. Therefore we arrive at the invalidation of uncertainty relation for the external time and energy by Aharonov and Bohm: the energy measurement can be reproducible and be made within an arbitrarily short time and within arbitrarily precision.

One interpretation of the time-energy uncertainty relation worth mentioning says that the time interval Δt in the uncertainty relation is the time for which the quantum system lasts unperturbed. This interpretation is to some extent supported by some facts in spectroscopy. For example, excited states have a finite lifetime and do not have a definite energy. The energy emitted by the excited states each time displays slight difference. The average energy of the photon released shows a peak value at the theoretical energy range of the state, but the distribution has a width called the natural linewidth. Slow-decaying states shows a narrow linewidth, which enables researchers to measure accurately the energy; whereas fast-decaying states shows a broad linewidth, which makes it difficult to measure the energy within a high precision (therefore microwave cavities are employed to slow down the decay-rate so that researchers may be able to get sharper peaks[50]) and to measure the rest mass of fast-decaying particles in high energy physics. The faster the particle decays, the less certain is its mass.

11 Mandelstam-Tamm uncertainty relation for time and energy

Mandelstam and Tamm presented a widely accepted time-energy uncertainty relations, which do not refer to event times but to characteristic times pertaining to the generic observables A. Let A be a non-stationary dynamical variable depending explicitly on time and thus can be applied to indicate time. The Heisenberg motion equation for $\hat{A}$ then reads

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \hat{A}\hat{H} - \hat{H}\hat{A}. \tag{11.1}$$

Using Robertson uncertainty relation (10.2), with a minor change of notation, we obtain the inequality

$$\Delta_\rho A \Delta_\rho H \geq \frac{1}{2}|\langle \hat{A}\hat{H} - \hat{H}\hat{A}\rangle_\rho|, \tag{11.2}$$

where $\langle X\rangle_\rho = \mathrm{tr}[\rho X]$, $(\Delta_\rho X)^2 = \langle X^2\rangle_\rho - \langle X\rangle_\rho^2$. Define a characteristic time by

$$\tau_\rho(A) = \frac{\Delta_\rho A}{\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{A}\rangle_\rho\right|}, \tag{11.3}$$

and combine it with (11.2), we have

$$\tau_\rho(A)\Delta_\rho H \geq \frac{1}{2}\hbar. \tag{11.4}$$

Here $\tau_\rho(A)$ is the time duration within which a change of A by ΔA occurs with a rate of $\mathrm{d}\langle \hat{A}\rangle/\mathrm{d}t$. Take the case of a free particle for example. Let A be

x, the particle position, therefore the denominator in Eq. (11.3) is the time derivative of the expectation value of position, i.e., the average of velocity,

$$\frac{\mathrm{d}\langle x\rangle_\rho}{\mathrm{d}t} = \frac{\langle \hat{p}\rangle_\rho}{m} = \langle \hat{v}\rangle_\rho. \tag{11.5}$$

Then the characteristic time has the following form

$$\tau_\rho(x) = \frac{\Delta_\rho x}{\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{x}\rangle_\rho\right|}, \tag{11.6}$$

which means that $\tau_\rho(x)$ is the time period during which the wave packet propagates by a distance which is equal to its width $\Delta_\rho x$. It can be interpreted that this is approximately the time it takes the packet to pass a given spatial point or that a period of time $\tau_\rho(x)$ is requied for the event 'the particle crosses a spatial point' to have a perceptible probability.

Besides the uncertainty relation involving characteristic time, Mandelstam and Tamm derived another uncertainty relation associated with lifetime. Let ψ_0 be a unit vector in Hilbert space and P a projection defined by

$$P := |\psi_0\rangle\langle\psi_0|. \tag{11.7}$$

The evolution of the quantum state along time is described in terms of the unitary operator $U_t = \exp(-\mathrm{i}tH/\hbar)$ as,

$$\psi_t = U_t\psi_0 = \mathrm{e}^{-\mathrm{i}tH/\hbar}\psi_0. \tag{11.8}$$

Substituting P for A into the Mandelstam-Tamm uncertainty relation yields

$$\Delta_{\psi_t} H \frac{\Delta_{\psi_t} P}{\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle P\rangle_{\psi_t}\right|} \geq \frac{\hbar}{2}. \tag{11.9}$$

Using $P^2 = P$, we obtain

$$\begin{aligned}
\Delta_{\psi_t} P &= \sqrt{\langle P^2\rangle_{\psi_t} - \langle P\rangle_{\psi_t}^2} \\
&= \left(\langle\psi_t|P|\psi_t\rangle - \langle\psi_t|P|\psi_t\rangle^2\right)^{1/2} \\
&= \left(\langle\psi_0|U_t^\dagger P U_t|\psi_0\rangle - \langle\psi_0|U_t^\dagger P U_t|\psi_0\rangle^2\right)^{1/2} \\
&= [\mathfrak{P}(1-\mathfrak{P})]^{1/2},
\end{aligned} \tag{11.10}$$

where

$$\mathfrak{P}(t) := \langle\psi_0|U_t^\dagger P U_t|\psi_0\rangle. \tag{11.11}$$

Note that

$$\begin{aligned}
\Delta_{\psi_t} H &= \sqrt{\langle H^2\rangle_{\psi_t} - \langle H\rangle_{\psi_t}^2} \\
&= \left(\langle\psi_t|H^2|\psi_t\rangle - \langle\psi_t|H|\psi_t\rangle^2\right)^{1/2} \\
&= \left(\langle\psi_0|U_t^\dagger H^2 U_t|\psi_0\rangle - \langle\psi_0|U_t^\dagger H U_t|\psi_0\rangle^2\right)^{1/2} \\
&= \left(\langle\psi_0|H^2 U_t^\dagger U_t|\psi_0\rangle - \langle\psi_0|H U_t^\dagger U_t|\psi_0\rangle^2\right)^{1/2} \\
&= \left(\langle\psi_0|H^2|\psi_0\rangle - \langle\psi_0|H|\psi_0\rangle^2\right)^{1/2} \\
&= \Delta_{\psi_0} H
\end{aligned} \tag{11.12}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle P\rangle_{\psi_t}\right| = \left|\frac{\mathrm{d}\mathfrak{P}}{\mathrm{d}t}\right|, \tag{11.13}$$

we obtain the inequality from the Mandelstam-Tamm relation (11.9),

$$\left|\frac{\mathrm{d}\mathfrak{P}}{\mathrm{d}t}\right| \leq \frac{2}{\hbar}\Delta_{\psi_0} H[\mathfrak{P}(1-\mathfrak{P})]^{1/2}. \tag{11.14}$$

Using the initial condition

$$\begin{aligned}
\mathfrak{P}(0) &= \langle\psi_0|U_0^\dagger P U_0|\psi_0\rangle \\
&= \langle\psi_0|P|\psi_0\rangle \\
&= \langle\psi_0|\psi_0\rangle\langle\psi_0|\psi_0\rangle \\
&= 1
\end{aligned} \tag{11.15}$$

and integrating the inequality (11.14), we have

$$\mathfrak{P}(t) \geq \cos^2(t\Delta_{\psi_0} H/\hbar), \qquad 0 \leq t \leq \frac{\pi}{2}\frac{\hbar}{\Delta_{\psi_0} H} \equiv t_0. \tag{11.16}$$

Giving the definition of the lifetime τ_P for P by

$$\mathfrak{P}(\tau_P) = \frac{1}{2}, \tag{11.17}$$

substituting τ_P for t into the inequality (11.16)

$$\frac{1}{2} \geq \cos^2(\tau_P \Delta_{\psi_0} H/\hbar) \tag{11.18}$$

and solving it, we finally arrive at the special version of Mandelstam-Tamm uncertainty relation for the case of projection $P = |\psi_0\rangle\langle\psi_0|$,

$$\tau_P \Delta_{\psi_0} H \geq \frac{\pi\hbar}{4}. \tag{11.19}$$

12 Bohr-Wigner uncertainty relation for time and energy

Fourier transform provides any wave propagation with uncertainty relations. It gives a complementary relation between the wavelength and the wave number as well as that between the period and the frequency. Based on the spirit of analogy, Bohr[51] proposed an uncertainty relation for time and energy, which seemed to correspond to the uncertainty relation for position and momentum,

$$\Delta t \Delta E \gtrsim h, \qquad \Delta x \Delta p \gtrsim h. \tag{12.1}$$

A careful discussion is due to Hilgevoord[52] on justifying a treatment of energy and time on equal footing to momentum and position.

Wigner carried out a more formal investigation into this topic. He considered a distribution for $t \geq 0$ concerning with the wave function ψ:

$$\mathfrak{P}_{x_0}(t) = \langle\psi_t|P_{x_0}|\psi_t\rangle = \langle\psi_t|x_0\rangle\langle x_0|\psi_t\rangle = |f(t)|^2, \tag{12.2}$$

where $f(t) = \langle x_0|\psi_t\rangle = \psi(x_0, t)$. The quantity $\Delta_f t$ measures the width of the distribution, while $\Delta_{\tilde{f}} E$ represents the width of $\tilde{f}$, the Fourier transform of f:

$$\tilde{f}(E) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{\mathrm{i}tE/\hbar} \mathrm{d}t. \tag{12.3}$$

Here we can generalize the method by substituting $|\phi\rangle$ for $|x_0\rangle$ and redefine f by

$$f(t) = \langle\phi|\psi_t\rangle. \tag{12.4}$$

The n-th moments of the probability distribution for time and energy are given by,

$$\langle t^n \rangle_f = \frac{\int_0^\infty |f(t)|^2 t^n \mathrm{d}t}{\int_0^\infty |f(t)|^2 \mathrm{d}t}, \tag{12.5}$$

$$\langle E^n \rangle_{\tilde{f}} = \frac{\int_0^\infty |\tilde{f}(E)|^2 E^n \mathrm{d}E}{\int_0^\infty |\tilde{f}(E)|^2 \mathrm{d}E}. \tag{12.6}$$

Combining the above expressions for moments and the variances of time and energy,

$$\Delta_f t = (\langle t^2 \rangle_f - \langle t \rangle_f^2)^{1/2}, \tag{12.7}$$

$$\Delta_{\tilde{f}} E = (\langle E^2 \rangle_{\tilde{f}} - \langle E \rangle_{\tilde{f}}^2)^{1/2}, \tag{12.8}$$

yields an uncertainty relation:

$$\Delta_f t \Delta_{\tilde{f}} E \geq \frac{\hbar}{2}. \tag{12.9}$$

For the time being, this is the purely mathematical consequence and its physical operational meaning still needs to be illustrated. Now consider a quantum state ψ prepared at $t = 0$ and assume that the measurement of energy is reproducible. Measure its energy of this state at $t > 0$, yielding a value in an interval Z which is centred at E_0 with the width being δE, i.e., $Z = (E_0 - \delta E/2,\, E_0 + \delta E/2)$, and then continue to measure the property $P_\varphi = |\varphi\rangle\langle\varphi|$. Provided that H has a non-degenerate energy spectrum, the probability for the event sequence is thus calculated as following:

$$\begin{aligned}
\mathfrak{P} &= \mathfrak{P}_\psi(E^H(Z), P_\phi) \\
&= \mathrm{tr}\left[P_\varphi E^H(Z) \mathrm{e}^{-\mathrm{i}tH/\hbar} |\psi\rangle\langle\psi| \mathrm{e}^{\mathrm{i}tH/\hbar} E^H(Z) P_\varphi\right] \\
&= \int_Z \mathrm{d}E \int_Z \mathrm{d}E' \langle\varphi|E\rangle\langle E|\psi\rangle\langle\psi|E'\rangle\langle E'|\varphi\rangle \mathrm{e}^{-\mathrm{i}t(E-E')/\hbar}.
\end{aligned} \tag{12.10}$$

Let the interval Z be small enough to ensure $\langle E|\varphi\rangle$ and $\langle E|\psi\rangle$ remain almost constant in Z. Then we obtain

$$\begin{aligned}
\mathfrak{P} &\cong |\langle E_0|\psi\rangle|^2 |\langle E_0|\varphi\rangle|^2 (\delta E)^2 \\
&\cong |\tilde{f}(E_0)|^2 (\delta E)^2.
\end{aligned} \tag{12.11}$$

In the basis on the approximation given above and in the case of H which has non-degenerate energy spectrum, we have

$$\begin{aligned} f(t) = \langle\psi_0|psi_t\rangle &= \int_{-\infty}^{\infty} \tilde{f}(E)\mathrm{e}^{-\mathrm{i}t(E-E')/\hbar}\mathrm{d}E \\ &\cong \mathrm{e}^{-\mathrm{i}tE_0/\hbar-|t|\Gamma/2\hbar} \end{aligned} \tag{12.12}$$

and

$$\tilde{f}(E) = |\langle E|\psi_0\rangle|^2 \cong \frac{1}{\pi}\frac{\Gamma/2}{(E-E_0)^2+(\Gamma/2)^2}. \tag{12.13}$$

Since there does not exist finite variance for the distribution $\tilde{f}(E)$, we may take $\delta E = \Gamma$ as the alternative of energy spread. Define

$$\mathfrak{P}(\tau) := \mathrm{e}^{-\tau\Gamma/\hbar}. \tag{12.14}$$

Using $\mathfrak{P} = |f(t)|^2$ and (12.12), we have

$$\mathrm{e}^{-\tau\Gamma/\hbar} = 1/\mathrm{e}, \tag{12.15}$$

and hence the relation for linewidth and lifetime

$$\tau\Gamma = \hbar. \tag{12.16}$$

Note that here the Wigner variances of time (12.7) and of energy (12.8) yield the results respectively:

$$\Delta_f t = \frac{\hbar}{2\Gamma} = \sqrt{2}\tau, \tag{12.17}$$

$$\Delta_{\tilde{f}} E = \Gamma/2, \tag{12.18}$$

the uncertainty relation of Wigner version thus reads

$$\Delta_f t\Delta_{\tilde{f}} E = \frac{\sqrt{2}}{2}\hbar. \tag{12.19}$$

In a new application[53] of Wigner time-energy uncertainty relation, $\tilde{f}(E)$ is identified as the energy amplitude and $|f(t)|^2$ is verified as equivalent to Kijowski's distribution for arrival time.

13 Summary and conclusion

We summarise the main topics involved in this work:

(1) In Section 1 we begin from the notions of time as both a parameter and a dynamical quantity and introduce the question how to formulate time into the quantum formalism. However Pauli pointed out the impossibility of defining a self-adjoint operator for time by means of whatever quantization rules. As to overcome Pauli's argument, physicists have proposed approaches that can be classified as three types: mathematical investigation into the argument itself, operational understanding of arrival-time distributions and incorporation of the time observable into quantum formulation.

(2) In Section 2 we consider the classical emsemble described by Liouville's equation, derive the first arrival-time current desity and connect it with the notion of absorption boundary. We also use stochastic process to show the n-th crossing equation based on the first crossing.

(3) After the introduction to classical arrival-time distribution, in Section 3 we list a series of important contributions due to some authors on incorporation of arrival time as a quantum observable into quantum formulation, such as the maximally symmetrized, albeit non-self-adjoint, arival-time operator by Aharonov and Bohm, Allcock's detection model based on absorbing potential, Kijowski's quantum arrival-time distribution translated from classical version, positive operator valued measures, etc..

(4) In section 4, we present the notion of quantum probability flux, which takes a different form from its classical counterpart, and emphasize that unlike the classical case this flux can be negative–the so-called backflow effect. Therefore we have to look for other well-defined positive probability flux.

(5) Then we introduce one of the most important positive distributions, Kijowski's distribution, and the axiomatic approach to derive it. We also point out the equivalence between Kijowski's distribution for positive momenta and that calculated under POVMs as well as the coincidence between the average arrival-time derived from Kijowski's distribution and that computed with the quantum flux.

(6) In Section 6 we present Aharonov-Bohm arrival-time operator and discuss in detail the alternative forms in momentum representation and, in particular, the non-orthogonality and completeness of its eigenvectors, leading to the definition of POVM. As an example of application, we employ the POVM theory to construct an arrival-time distribution and confirm its coincidence with Kijowski's distribution.

(7) In Section 7 we first review the classical notion of local density and then list some candidates for its quantization with symmetrized forms. We also show that different candidates imply different latent properties, and that, as an example, two velocity density operators lead to Kijowski's distribution and quantum probability flux respectively.

(8) As another important example, we introduce Halliwell-Yearsley arrival-time distribution, which is an almost positive distribution under the condition of small complex potentials V_0 and the long-term assumption $V_0\tau \gg 1$. we use these authors' path integral approach, together with the small V_0 assumption, to derive the distribution expression, which turns out to be in terms of the quantum flux.

(9) In Section 9, we give a short introduction to some dynamical times such as phase times, tunneling times and transmission times.

(10) In the rest part of this work we continue to discuss time-energy uncertainty relation. We start from deriving the Robertson uncertainty relation for general Hermitian operators and point out that due to lack of self-adjoint time operator this relation does not apply to energy and

time. Then we introduce the different ideas of interpreting the meaning of Δt in the time-energy uncertainty relation demanded. Of those interpretations, the version regarding time as the external parameter is easily discarded, but the interpretation is also invalidated by Aharonov and Bohm that views Δt as the time duration required to measure energy with a given resolution of ΔE or with an energy disturbance ΔE exerted inevitably by measurement. Finally we mention an idea that suggest Δt be the time interval for which the quantum system exists unperturbed.

(11) In Section 11 we introduce the notion of Mandelstam-Tamm uncertainty relation involving characteristic time and lifetime respectively.

(12) Finally we discuss Bohr-Wigner time-energy uncertainty relation based on Fourier transform and, in particular, Wigner's formal mathematical calculation combined with its physical operational interpretation.

To understand the various aspects of time in quantum mechanics has been a conceptual challenge, as is shown in this work, wherein a few important and representative theoretical approaches are reviewed. We have paid particular attention to attempts and progress made in the topics of arrival time and time-energy uncertainty relations.

For the former topic, a great many achievements have been made to overcome a number of obstacles. For example, many authors disregard Pauli's argument on the non-existence of the self-adjoint time operator by realizing that an observable is not necessarily associated with a self-adjoint operator, and circumvent it by constructing specific arrival time operators that yield required properties of time; and negative conclusion put forward by Allcock has also been opposed. In addition, many results have been established to incorporate the observable of time into quantum formulation by means of arrival-time distribution; and it is commonly realized nowadays that arrival time distribution is one of the most important and promising tools to investigate into the nature of time.

For the latter topic, i.e., time-energy uncertainty relations, there are a number of interpretations subject to debate, of which, in particular, the intrinsic times of quantum systems are identified to most satisfactorily respect the uncertainty relation with energy without violating the existing principles of quantum theory or postulating additional independent principles.

By and large, although there are still many issues awaiting to be studied, we have undertaken much fruitful investigation into the topics concerning time in quantum theory and made great progress over the past century.

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